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CONTRIBUTIONS FROM THE JEFFERSON PHYSICAL LABORATORY, HARVARD UNIVERSITY.

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The desirability of an electromagnetic theory of gravitation has been pointed out many times, by Maxwell,[1] Lorentz,[2] and others, who would assume that gravitation is due to a slight excess in the attraction of an electric charge for one of the opposite sign over its repulsion for an equal one of the same sign; and also by Einstein and his followers, who would have changes of gravitational attraction propagated with the velocity of light. But Maxwell and Lorentz had no proof of the truth of their theory, or reason for belief in it, other than that it might connect gravitation and electricity; and even Einstein had no proof, but only the fact that if the velocity were different from that of light, we should have a means of detecting absolute motion. The object of this paper is to give some reasons for belief in the electromagnetic theory, and to point out some of its most surprising results.

Objections have been raised against such a theory, principally on the ground that it would involve gravitational aberration like the aberration of light, but I hope to show that the electromagnetic theory involves no such consequences. Wiechert[3] and others have also objected on the ground that gravitation is essentially different from electrostatic force, especially in its ability to penetrate even the densest matter without appreciable change. But it may be proved, with no other assumption than that a dielectric affects electrostatic force only through the displacement of negatively charged particles within it, that the uniform gravitational permeability of all dielectrics is no argument against the electromagnetic theory. And it has been proved by Gans[4] that the equality of the gravitational permeabilities of vacuum and conductors is also consistent with such a theory.

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[1] Maxwell, Electricity and Magnetism, Vol. 1, p. 40.

[2] Versl. K. Ac. van Wet., **8**, 603 (1900); **8**, 616 (1900).

[3] Wiechert, Über die Relativitätsprincip und Äther, Phys. Zeitsch., **12**, 18–19.

[4] Phys. Zeitsch., **6**, 803.

Notation. — The notation used in this paper will be that of Gibbs, in which all vectors may be distinguished by being printed in Clarendon type, while scalars will be in italic type. Thus, **a** would be a vector, and a a scalar. The magnitude of any vector **a** will be denoted by $|\mathbf{a}|$.

The scalar product,

$$(\mathbf{a}_x\mathbf{b}_x + \mathbf{a}_y\mathbf{b}_y + \mathbf{a}_z\mathbf{b}_z),$$

of two vectors **a** and **b** will be denoted by $\mathbf{a}\cdot\mathbf{b}$; and their vector product,

$$\mathbf{i}\,(\mathbf{a}_y\mathbf{b}_z - \mathbf{a}_z\mathbf{b}_y) + \mathbf{j}\,(\mathbf{a}_z\mathbf{b}_x - \mathbf{a}_x\mathbf{b}_z) + \mathbf{k}\,(\mathbf{a}_x\mathbf{b}_y - \mathbf{a}_y\mathbf{b}_x),$$

by $\mathbf{a}\times\mathbf{b}$, where **i**, **j**, and **k** are the unit vectors in the directions of x, y, and z, respectively.

The symbol ∇ will be used for the operator,

$$\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right),$$

so that ∇a is the gradient of scalar a, a vector; $\nabla\cdot\mathbf{a}$ is the divergence of vector **a**, a scalar; and $\nabla\times\mathbf{a}$ is the curl of vector **a**, another vector.

The "mass" of a body must be understood to mean its mass as measured on a system of axes on which it is at rest; and similarly the density, which will be denoted by the letter ρ, will mean the limit of the ratio of the mass in a volume element to the volume of the element, with the above interpretation of the word mass. The mass of a body will, therefore, be a measure of the amount of matter, or "gravitational charge" in the body, which will be different for moving bodies from the "inertia" usually understood by the word mass.

Other Theories of Gravitation. — With this notation we may now examine the results of various methods by which we might imagine the changes of gravitational force to be propagated.

First, let us imagine, according to what I shall call "Theory I," that the changes of gravitational force due to any changes in the position of matter take place at the same instant at every point in space. In this case, if **g** is the gravitational force per unit mass at any point,

$$\mathbf{g} = k\nabla\int\!\!\int\!\!\int_{\infty}\frac{\rho}{r}\,d\tau = -k\int\!\!\int\!\!\int_{\infty}\frac{\rho}{r^2}\,\mathbf{r}_1 d\tau$$

where r is the distance from the volume element $d\tau$ to the point at which the integral is to be evaluated, $\mathbf{r}_1$ the unit vector in the direc-

tion from the element, and k the gravitation constant, 6.480×10^{-8} $\frac{\text{cm}^3}{\text{gm. sec.}^2}$. This assumption involves the existence of "absolute time" and instantaneous action at a distance, but is a possible hypothesis if these are possible.

The hypothesis which I shall call "Theory II" is the consequence of most of the mechanical explanations of gravitation, such as the well known Le Sage theory, and theories of attraction through the pressure of invisible radiation or through the pulsations of slightly compressible electrons in a less compressible fluid. According to this theory the attraction at any time t on any particle is determined by the matter that occupied each volume element at a time $\left(t - \frac{r}{C}\right)$, where C is some very large velocity, and r is, as above, the distance from the element. In this case, if we let $[\rho]$ equal the value of ρ in the element at this time, we have

$$\mathbf{g} = -k \int\int\int_{\infty} \frac{[\rho]}{r^2} \mathbf{r}_1 d\tau.$$

For "Theory III" let us suppose any particle moving uniformly to carry a perfectly symmetrical system of lines of force with it, so that at any instant the forces due to it at all equidistant points will have equal intensities and be directed exactly towards it. But if its motion is changed by an infinitesimal amount, let us suppose that the disturbance of the force is propagated with a velocity C relative to that of the particle at the time the change of its velocity takes place.

In this case,

$$\mathbf{g} = -k \int\int\int_{\infty} \frac{[\rho]'}{r'^2} \mathbf{r}_1' d\tau,$$

where r' is the distance at which the matter would have been at time t, that occupied the element $d\tau$ with density $[\rho]'$ at the time $\left(t - \frac{r'}{C}\right)$, if it had kept since that time the velocity it then had. We may also say

$$\mathbf{g} = -k \int\int\int_{\infty} \frac{[\rho]\mathbf{r}_1' d\tau}{r^2(1 + \mathbf{B}^2 - 2\mathbf{B} \cdot \mathbf{r}_1)^3},$$

where $\mathbf{B}$ is the ratio of the velocity of the matter in the element $d\tau$ at the time $\left(t - \frac{r}{C}\right)$ to the velocity C.

For "Theory IV" let us assume that there is another vector $\mathbf{h}$ related to the vector $\mathbf{g}$ in the same way that magnetic force is related to electric, and that these vectors satisfy the set of equations:

$$\nabla \cdot \mathbf{g} = -4\pi\rho k$$

$$\nabla \cdot \mathbf{h} = 0$$

$$\nabla \times \mathbf{h} = \frac{\partial \mathbf{g}}{\partial(Ct)} - 4\pi\rho k\mathbf{B}$$

$$\nabla \times \mathbf{g} = -\frac{\partial \mathbf{h}}{\partial(Ct)}$$

$$\mathbf{f} = \mathbf{g} + \mathbf{B} \times \mathbf{h},$$

where $\mathbf{f}$ is the force per unit mass due to the gravitational force $\mathbf{g}$ and "gravimagnetic" force $\mathbf{h}$. It will be noticed that these equations are very much like the electromagnetic equations:

$$\nabla \cdot \mathbf{E} = 4\pi\epsilon$$

$$\nabla \cdot \mathbf{H} = 0$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{E}}{\partial(ct)} + 4\pi\epsilon\boldsymbol{\beta}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial(ct)}$$

$$\mathbf{F} = \mathbf{E} + \boldsymbol{\beta} \times \mathbf{H}$$

where $\mathbf{E}$ and $\mathbf{H}$ are electric and magnetic forces, ϵ the charge per unit volume, $\boldsymbol{\beta}$ the ratio of the velocity of the charge at any point to the velocity, c, of propagation of electromagnetic disturbances, and $\mathbf{F}$ the force per unit charge due to electric and magnetic forces.

If we assume with Einstein that there is no ether, and no universal or absolute time, we readily see that these theories are all inconsistent with this belief, with the single exception of the special case of Theory IV which we may call "Theory V," in which $C = c$. If, on the other hand, we assume the "conditional relativity principle," of Wiechert [5] and assume the existence of ether and absolute time, we see a wider range for Theory IV than that obtained by merely giving different values to C, in that Theory IV does not define any particular state of motion that may be assumed to be the state of absolute rest, and un-

[5] Wiechert, Relativitätsprinzip und Äther, Phys. Zeitsch., **12**, 18–19.

less we make such a definition we have no single-valued definition of any of our time derivatives, even though we have a definition of absolute time. The physical significance of this statement is that, according to Theory IV, all gravitational disturbances must be propagated with a velocity C through an ether that is not necessarily at rest relative to the electromagnetic ether. Therefore to define Theory V completely in terms of Theory IV, we must say that the two ethers must be at rest relative to each other, and that the two velocities C and c must be equal.

Objection against Theories I–III. — If we now examine further the consequences of these theories, we shall see that Theory I, if not impossible, is highly improbable; and that Theory V is the only other that does not involve a continual increase of energy of any system of gravitating bodies.

Assuming the unconditional relativity principle, all but Theory V are, of course, impossible. But assuming the conditional relativity principle, we see that Theory I involves either instantaneous action through a medium that is also capable of propagation of disturbances with a finite velocity, or else it involves true "action at a distance." Also it has been shown by Wilkens[6] that the changes of the perihelion of Mercury and other astronomical phenomena, which are consistent with the electromagnetic theory, are inconsistent with Theory I.

These difficulties make it necessary to look for some other theory, such as II. But if we assume the truth of this theory, we must say that the components of a binary star revolving in circles about their centre of gravity will each be accelerated at time t by a force directed towards a point where the other was at time $\left(t - \frac{r}{C}\right)$; and this force will have a component in the direction of motion which is an infinitesimal, if C is allowed to become infinite, of the order of $|\,\mathbf{B}\,|$ for the other body. But no matter how large C may be, this component of the force will never be absolutely zero. Hence the ever increasing kinetic energy would tear any such system apart, even though the components were unequal and the orbits not circular; and even a rotating planet would increase its energy of rotation until it burst, and would continue increasing its energy forever, even after disruption; so that, if Theory II held, the earth and the whole solar system would have gone to pieces long ago and left not even the law of the conservation of energy for a vain consolation.

Theory III might be expected to remove this difficulty, but it does

[6] Phys. Zeitsch., 7, No. 23, 846.

not, though it does reduce the component of force in the direction of motion to an infinitesimal of the third order. For if we consider a system where A_1 and A_2 are the positions of the two masses m_1 and m_2, revolving in circular orbits at the time t, B_1 and B_2 their positions at the time $\left(t - \frac{C_1A_2}{C}\right)$ and $\left(t - \frac{C_2A_1}{C}\right)$, and C_1 and C_2 the positions they would have occupied if they had kept since the above-mentioned times the velocities they then had, we see that C_1 and B_1 must always lie on the same side of the line A_2A_1 produced. If the velocity C becomes infinite, the distances from C_1 and C_2 to the line A_1A_2 are obviously infinitesimals of the third order, but they can never be exactly zero; and hence there will always be a forward component of force that must ultimately produce the same disastrous results that we have seen in Theory II. Hence we see that Theories II and III are almost certainly false.

Theory IV. — Let us now suppose that Theory IV is correct, and examine its consequences. By Green's Theorem it may easily be proved that the gravitational energy liberated in scattering any distribution of gravitating matter to infinity is

$$-\frac{1}{8\pi k}\int_{\infty}\int\int \mathbf{g}^2\, d\tau$$

if the matter is all at rest. But if the matter is in motion, we must add to this, to get the gravitational energy liberated in bringing it all to rest at infinity,

$$-\frac{1}{8\pi k}\int_{\infty}\int\int \mathbf{h}^2\, d\tau.$$

And if the ratio of the velocity of matter at every point to that of light is the vector point function $\boldsymbol{\beta}$, the total kinetic energy in the universe is

$$\int_{\infty}\int\int \rho c^2\,(R^{-1} - 1)\, d\tau$$

where $R = \sqrt{1 - \boldsymbol{\beta}^2}$. We may now assume as in the corresponding electromagnetic case, that the integrands in the above expressions actually represent the energy that would be removed in the scattering process from the volume elements for which they are evaluated, and

hence that the energy in a region T of any distribution of matter acted upon only by gravitational forces is

$$\int\int\int_T \{\rho c^2 (R^{-1} - 1) - \frac{1}{8\pi k} (\mathbf{g}^2 + \mathbf{h}^2)\} \, d\tau.$$

While this assumption is not necessarily true for any finite region, it is certainly true in the limit for a sphere whose radius is allowed to become infinite.

Gravitational Radiation. — We may prove the following THEOREM:

If a distribution of matter is affected by forces of which none but those of gravitational origin do any work, the energy, E_s, within any closed surface, S, which neither matter nor electromagnetic energy enters or leaves, and which has a normal at every point, will increase or decrease at such a rate that

$$\frac{dE_s}{d(Ct)} = + \frac{1}{4\pi k} \int\int_S \mathbf{g} \times \mathbf{h} \cdot \mathbf{dS}$$

where $\mathbf{dS}$ is the element of surface considered as a vector in the direction of the exterior normal.

The truth of this theorem depends on that of the above assumption about the energy in the region T, but it is certainly true for the infinite sphere.

The proof (like that of Poynting's theorem), is as follows: If T is the space within S,

$$E_s = \int\int\int_T \{\rho c^2 (R^{-1} - 1) - \frac{1}{8\pi k} (\mathbf{g}^2 + \mathbf{h}^2)\} \, d\tau$$

$$\frac{dE_s}{d(Ct)} = \int\int\int_T \left\{ \frac{\partial}{\partial (Ct)} [\rho c^2 (R^{-1} - 1)] - \frac{1}{4\pi k} \left(\mathbf{g} \cdot \frac{\partial \mathbf{g}}{\partial (Ct)} + \mathbf{h} \cdot \frac{\partial \mathbf{h}}{\partial (Ct)} \right) \right\} d\tau$$

$$= \int\int\int_T \left\{ \frac{\partial}{\partial (Ct)} [\rho c^2 (R^{-1} - 1)] - \frac{1}{4\pi} (\mathbf{g} \cdot \nabla \times \mathbf{h} - \mathbf{h} \cdot \nabla \times \mathbf{g}) - \rho \mathbf{g} \cdot \mathbf{B} \right\} d\tau.$$

But $\rho\mathbf{g}\cdot\mathbf{B}$ is the increase of kinetic energy per unit volume per unit Ct for a volume element moving with the matter in it; hence

$$\frac{dE_s}{d(Ct)} = \frac{1}{4\pi k}\int\int_T\int (\mathbf{h}\cdot\nabla\times\mathbf{g} - \mathbf{g}\cdot\nabla\times\mathbf{h})\, d\tau$$

$$= \frac{1}{4\pi k}\int\int_T\int \nabla\cdot(\mathbf{g}\times\mathbf{h})\, d\tau$$

$$= \frac{1}{4\pi k}\int\int_T\int \mathbf{g}\times\mathbf{h}\cdot\mathbf{dS} \qquad \text{Q. E. D.}$$

To find the amount of negative gravitational energy radiated from an accelerated particle when it is at rest, we may make use of the result of the corresponding electrical case. In this case, if the charge on the electron is e, and the accleration is $\mathbf{a}$, and the unit vector in the direction from the electron to the point considered is $\mathbf{r}_1$, the electric force due to radiation at a distance r is

$$\mathbf{E} = -\frac{e}{c^2 r}\mathbf{a}_p,$$

where $\mathbf{a}_p$ is the component of $\mathbf{a}$ perpendicular to $\mathbf{r}_1$, and the magnetic force due to acceleration is

$$\mathbf{H} = +\frac{e}{c^2 r}\mathbf{a}\times\mathbf{r}_1 = \mathbf{r}_1\times\mathbf{E}.^{7}$$

The magnitudes of these forces are seen to be equal, and the directions at right angles. For the corresponding gravitational case, we need only to change the signs of these expressions and substitute mk for e, and C for c, and we have

$$\mathbf{g} = \frac{mk}{C^2 r}\mathbf{a}_p,$$

$$\mathbf{h} = -\frac{mk}{C^2 r}\mathbf{a}\times\mathbf{r}_1 = \mathbf{r}_1\times\mathbf{g}.$$

We now see that

$$\mathbf{g}\times\mathbf{h} = \frac{m^2k^2\mathbf{r}_1}{C^4r^2}\mathbf{a}_p{}^2 = \frac{m^2k^2}{C^4r^2}(\mathbf{a}\times\mathbf{r}_1)^2\mathbf{r}_1.$$

[7] For proof see Lorentz, "Theory of Electrons," Chapter 1.

But if we now have two particles with constant accelerations, oppositely directed, we see that the vectors $\mathbf{g}$ and $\mathbf{h}$ for one of the particles will have nearly opposite directions to the corresponding vectors for the other particle at points very far from the two masses. For such points we may then say the total gravitational force is

$$\begin{aligned}\mathbf{g} &= \frac{m_1 k}{C^2 r_1}\mathbf{a}_{p1} + \frac{m_2 k}{C^2 r_2}\mathbf{a}_{p2} \\ &= \frac{k}{C^2}\cdot\frac{m_1 r_2 \mathbf{a}_{p1} + m_2 r_1 \mathbf{a}_{p2}}{r_1 r_2} \\ &= \frac{k m_1}{C^2 r_1 r_2}\left(r_1 + \frac{\mathbf{a}_{p2}}{\mathbf{a}_{p1}}\cdot\frac{m_2}{m_1}r_2\right)\mathbf{a}_{p1}.\end{aligned}$$

The fraction $\mathbf{a}_{p2}/\mathbf{a}_{p1}$ will have a meaning only in a case where it is multiplied by a product of a scalar by $\mathbf{a}_{p1}$, or in a case where $\mathbf{a}_{p1}$ and $\mathbf{a}_{p2}$ have the same direction, as in the limit $r_1 = \infty$, when it will approach a scalar limit. Unless this limit is $-m_1/m_2$, $\mathbf{g}$ will be an infinitesimal of the order of $1/r_1$. But if $\lim \mathbf{a}_{p2}/\mathbf{a}_{p1} = -m_1/m_1$, then we readily see that $\mathbf{g}$ and $\mathbf{h}$ are both infinitesimals of the order of $1/r_1^2$ or of a higher order, and $\mathbf{g}\times\mathbf{h}$ is of the order of $1/r_1^4$ or higher. But this is possible for all directions only if $m_1\mathbf{a}_1 = -m_2\mathbf{a}_2$, and therefore in this case, and only in this case,

$$\lim_{r=\infty}\iint_S \mathbf{g}\times\mathbf{h}\cdot d\mathbf{S} = 0$$

where S is a sphere of radius r, surrounding the particles. Hence we see that there is no radiation of negative energy from a pair of uniformly accelerated particles momentarily at rest, if, and only if,

$$m_1\mathbf{a}_1 = -m_2\mathbf{a}_2.$$

Radiation from a Rotating Body. — Since Theory IV, with $C \neq c$, is inconsistent with the unconditional relativity principle, we must, in assuming the truth of it, assume the existence of an ether and an absolute system of simultaneity. But if now we assume that gravity is transmitted through an ether that is independent of the electromagnetic ether, and moving through it, we must assume the time to be the same absolute time in both cases, and the lengths to be measured in absolute units of length.

If, with these assumptions, we consider the motion of a body rotating,

undisturbed by external influences, about an axis of symmetry that is at rest in the gravity ether, we may determine whether it can satisfy the condition

$$\frac{dE}{d(Ct)} = 0.$$

It might now be supposed that we could not do this by determining whether

$$m_1\mathbf{a}_1 = -\, m_2\mathbf{a}_2,$$

because the radiation from an accelerated electron already in motion is not the same as if it had the same acceleration when at rest. But it is obvious that, if we consider any direction making given angles with the directions of the velocity and acceleration, the ratio of the rate of radiation in that direction in the case with velocity to the corresponding rate in the case with no velocity is a function of only the magnitude of the velocity and the angle between the velocity and acceleration, and not a function of the charge or acceleration of the electron. Hence we see that, if we consider a case where the two variables that determine this ratio are the same for every point of the infinite sphere, we may then, and only then, say that there is no radiation of negative energy from a pair of uniformly accelerated gravitating particles if, and only if,

$$m_1\mathbf{a}_1 = -\, m_2\mathbf{a}_2.$$

Returning to our rotating body, which may be any solid, liquid, gas, or collection of very minute particles like Saturn's rings, provided only that all points in any ring of infinitesimal cross-section around the axis shall have the same density and the same constant angular velocity around it, we see that the condition that no energy shall be radiated to infinity from the body is that there shall be no radiation to infinity from any of these rings.

First let us assume that the axis is stationary in the gravity ether and perpendicular to the direction of motion of the electric ether through it, and let us see when this condition will be fulfilled. For simplicity let us consider first only the case of a collection of particles such as Saturn's rings or a rotating nebula with no internal forces but those of gravity. In such a case it is obvious from symmetry that if the inertia of a particle were proportional to its mass and independent of its velocity, the system could rotate indefinitely, preserving always its symmetry about the stationary axis, and, since the accelerations at opposite points would be equal, opposite, and constant, not radiating

any energy. But since the inertias of two equal particles would be unequal when they were going in opposite directions not perpendicular to the direction of motion of the electric ether, we see that if the system were brought into the condition assumed above, the accelerations at opposite points would be unequal and there would be radiation. Hence to avoid radiation we must have the two ethers relatively at rest.

If we introduce internal forces other than those due to gravity, we make the problem much more complicated, but it is evident that the results would be of the same general nature, and that if the two ethers were relatively at rest there would be absolutely no radiation.

With this assumption of relative rest, we may now solve the problem of the radiation from such a collection of minute particles rotating about its axis, and at the same time moving through the ethers with a velocity comparable to that of light. From the similarity of the gravitational equations to the electromagnetic, we see that if we introduce a gravitational relativity principle, exactly like the electromagnetic relativity principle with C substituted for c in all formulas, the condition that two equal volume elements opposite to each other in the body shall not radiate negative energy to infinity is that the accelerations of the matter in them measured in the gravitational units of moving distance and local time shall be equal and opposite.

We may now suppose the particles to be brought into positions and velocities that appear at a certain instant of the gravitational local time of a system moving with the axis to be absolutely symmetrical about the axis. But since the equation "force equals rate of increase of momemtum," holds only when the electromagnetic units of local time and moving distance are used, we see that the accelerations of opposite particles are equal in the gravitational system only if all the corresponding units are equal, or if

$$C = c.$$

And we see also that if other internal forces are introduced, the problem is again more complicated, but that we have a similar result, that there is absolutely no radiation of negative energy from any body rotating about an axis of symmetry only with Theory V.

Radiation from two or more Gravitating Bodies. — If, however, we consider the more general case of two or more bodies moving under the action of no forces but those of gravity, we shall find a small amount of radiation even with Theory V. But this unavoidable radiation is very small, being in general for any pair of bodies less than the product

obtained by multiplying the radiation from one of them alone by the square of the ratio of their relative velocity to the velocity of light. And we shall prove that while, for small bodies, this radiation is due only to the changes in the accelerations in the time required for propagation of radiations from one to the other, Theory IV involves radiations that may be larger than this in any desired ratio, and that are due to actual differences of mass acceleration.

To prove this, let us determine, from the equations of Theory IV, the accelerations of two masses, m_1 and m_2, moving through the gravity ether with equal opposite velocities parallel to that of the electric ether, and whose ratios to C are $\mathbf{B}_1$ and $\mathbf{B}_2$, let the direction of x be that of $\mathbf{B}_1$ and let us suppose the two particles to be moving so that m_2 is in the direction of the axis of y, at a distance a from m_1, at the instant considered. If now we let r_1 be the distance of any point from m_1, in a direction making an angle θ_1 with that of x, we have the gravitational force due to m_1, neglecting that due to radiation and change of $\mathbf{B}_1$ in time r_1/C, directed towards m_1 at every point, and of intensity

$$| \mathbf{g}_1 | = \frac{km_1}{r_1^2} (1 - \mathbf{B}_1^2) (1 - \mathbf{B}_1^2 \sin^2 \theta_1)^{-\frac{3}{2}},$$

while the gravimagnetic force due to m_1 is

$$\mathbf{h}_1 = \mathbf{B}_1 \times \mathbf{g}_1.$$

These formulas may be proved by substitution in the equations of Theory IV, assuming $\mathbf{B}_1$ constant.

Therefore, neglecting radiated forces from m_1 and change of $\mathbf{B}_1$, the force acting on m_2 is

$$m_2 \mathbf{f}_2 = m_2 (\mathbf{g}_1 + \mathbf{B}_2 \times \mathbf{h}_1).$$

Therefore

$$m_2 \mathbf{a}_2 = - \mathbf{j} \frac{km_1 m_2}{a^2} R_2 (1 - \mathbf{B}_1^2)^{-\frac{1}{2}} (1 - \mathbf{B}_2 \cdot \mathbf{B}_1).$$

Similarly

$$m_1 \mathbf{a}_1 = + \mathbf{j} \frac{km_1 m_2}{a^2} R_1 (1 - \mathbf{B}_2^2)^{-\frac{1}{2}} (1 - \mathbf{B}_1 \cdot \mathbf{B}_2).$$

Since $\mathbf{B}_2 = - \mathbf{B}_1$

we have $m_1 \mathbf{a}_1 = - m_2 \mathbf{a}_2$

if, and only if, $R_1 = R_2$,

or $\boldsymbol{\beta}_1 = - \boldsymbol{\beta}_2$,

in which case the forces due to radiation are also equal and opposite.

Therefore to avoid continually increasing energy we must again have the two ethers at rest relative to each other.

Now let us consider a similar case where the centre of the line joining them is not at rest in the ether. Here we may again introduce the gravitational relativity principle and with the aid of the gravitational local time and auxiliary distance units thus defined, we may solve the problem exactly as we did the one above. But we now see that if the velocity of m_1 in the new system is in the direction of the velocity of the new system by the old, and if

$$\mathbf{B}_2' = -\mathbf{B}_1'$$

where primes denote measurement on the moving system, then, since the electromagnetic units of apparent distance and local time are different from the corresponding gravitational quantities,

$$\boldsymbol{\beta}_2' \neq \boldsymbol{\beta}_1', \quad \text{and} \quad R_2' \neq R_1'$$

unless

$$C = c.$$

It seems strange at first sight that Theory V is not overthrown by the same considerations that proved Theory III impossible even for the case of a rotating body. But the arguments used in that case do not apply here; because every particle in the body is affected at every instant by forces radiated from the other particles, and it is obvious from qualitative considerations that they will exert a backward force on the particle of the order of $|\boldsymbol{\beta}'^3|$ times the total force on it, and it is further evident from the fact that the radiation theorem proves that there is no gain of energy that this force must exactly balance the forward component of the non-radiational force.

Reasons for Preference of Electromagnetic Theory. — We now see that Theory V is the only theory we have examined, except the practically impossible Theory I, that does not involve an extremely improbable, continual increase of energy in any rotating planet, sun, or nebula. With such a continual increase of energy it is doubtful if the solar system could have been formed. Furthermore, Theory V is the one that involves the least radiation of negative energy, which we see is a destructive process, from such a system as the solar system is at present. Of course, we cannot prove that the solar system would yet have gone to pieces with more radiation than this, but we can say that if there had been enough additional radiation of this sort it would have.

We have, however, a much more convincing argument, for if we think of such laws as the law of the minimum possible loss of heat in electric

currents, the principle of "Least Work" so useful in civil engineering, the principle of "Least Action" or "Hamilton's Principle" in dynamics of matter and of electricity, and many other laws of this sort, it seems highly probable that, if there must be destructive radiation of gravitational energy, the laws by which such radiation takes place should be those involving the least amount of it.

Another argument of the same general nature is furnished by the extreme simplicity of the fundamental equations of natural phenomena, such as the equations of electrodynamics, either in Maxwell's form or in terms of scalar and vector potentials, the equation for the flow of heat in a solid, and many others, that all combine to make us believe that, if there is one set of equations for gravity that are simpler than all other possible equations not involving improbable results, these are the ones that express the phenomena correctly. This argument of simplicity is all the more striking if we write the electromagnetic equations in the remarkably simple form that they assume when expressed in the four-dimensional vector analysis of Lewis.[8] This analysis gives us a set of gravitational equations of equal simplicity, if, but only if, we adopt the electromagnetic theory. And for Theory V we have not only a similar set of equations, but, when we consider their meaning, a simpler explanation of their fundamental causes than can readily be found with any other theory. With these considerations combined with the experimental evidence furnished by the planet Mercury, it seems as though we could hardly help believing in this theory.

To see what Theory V means we have only to consider the theory that matter is made up wholly of electrons and the corresponding positive charges. If now we suppose every positive charge at rest to repel every other positive charge with a weaker force than that with which it attracts an equal negative charge in the same relative position, the ratio of the difference between these forces to the attractive force being a very small number G, we see that every particle of matter at rest will attract every other particle with exactly the observed force if G is properly chosen. And if now we suppose that the magnetic force from a moving positive charge has a smaller effect on another moving positive charge than on an equal negative one with the same relative position and velocity, and that the ratio of the difference between them to the force between those of opposite signs is the same number G, the existence of the vector $\mathbf{h}$ is explained. But we must assume forces between two negative charges to be equal and opposite to those between positive and negative, for, as Gans[9] has shown, this set of assumptions

[8] These Proceedings, **46**, No. 7, October, 1910.

[9] Gans, Phys. Zeitsch., **6**, No. 23, 803.

is the set which makes the gravitational permeability of any conductor equal to that of a vacuum. And we see, furthermore, that if electrostatic forces are to obey the principle of relativity, and the first of these assumptions is made, the second and the obedience of the force of gravity to the relativity principle are necessary consequences of the first.

By comparison of the electrostatic repulsion of two electrons, whose ratio of charge to inertia is 5.595×10^{-17}, with the gravitational attraction between two spheres, we would find that

$$G = 1.67 \times 10^{-42},$$

if the fundamental positive charge were similar to the negative electron.[10] This readily accounts for the fact that its existence has never been experimentally observed.

We may or may not, as we please, assume that this slight difference between the forces is produced by the same cause which produces the rest of the forces, but from the fact that it is transmitted through the same ether, with the same velocity, it seems most reasonable to assume that it is due to the same cause.

Consequences of this Theory, Negative Energy. — One of the most surprising facts about the electromagnetic theory of gravitation, and one with most surprising consequences, is the fact that we have to introduce the conception of negative energy, both potential and kinetic, distributed through the space around any gravitating matter. And by the term, "negative energy," we cannot mean a mere diminution of the positive energy in the space, as the following considerations will show.

First let us consider the energy radiated from an accelerated mass, whose inertia is balanced by the inertia of some mass at a great distance, so that the radiations of gravitational energy cannot interfere for a considerable time. We now see that during that time the transfer of energy across any element of surface $\mathbf{dS}$ per unit ct is

$$-\frac{1}{4\pi k}\mathbf{g}\times\mathbf{h}\cdot\mathbf{dS},$$

[10] Since the inertia of an electric charge e on a sphere of radius a is proportional to e^2/a, G is different from the above value if the fundamental positive charge has not the same e/a as the electron, where a in this case is a sort of average radius of points in the electron or positive charge. But the fact that k is the same for all kinds of matter indicates that, if there is only one kind of positive electricity, with only one value of G, it must be collected in fundamental charges for all of which e/a is the same. Hence we have reason to believe that all atoms of positive electricity are probably alike.

but that if we draw two surface elements, $\mathbf{dS}_1$ and $\mathbf{dS}_2$, parallel to each other and perpendicular to the same line of the vector $\mathbf{g}\times\mathbf{h}$ at a distance dx apart, we see that, if $\mathbf{dS}_2$ is in the direction of $\mathbf{g}\times\mathbf{h}$ from $\mathbf{dS}_1$, the energy radiated from a mass that is accelerated for an infinitesimal time crosses $\mathbf{dS}_2$ at a time $\frac{dx}{c}$ units of time later than that at which it crosses $\mathbf{dS}_1$, so that the transfer is actually in the direction of $\mathbf{g}\times\mathbf{h}$. But such radiated energy has lost all connection with its source, and there is no positive energy radiated, hence it is negative energy that is transferred in the direction of $\mathbf{g}\times\mathbf{h}$ across these surfaces.

It is also interesting to notice that in the case of a mass moving with uniform velocity the vector $\mathbf{g}\times\mathbf{h}$ has the same direction as the motion at all points in the plane through this mass perpendicular to this direction. This also indicates that the mass carries negative energy with it as it moves. But if the mass is not charged with electricity, the electric and magnetic forces due to the electrons and corresponding positive charges within it are zero at points outside the mass. Therefore, the electromagnetic energy is wholly within it, and the external gravitational energy cannot be considered as a mere diminution of the electromagnetic energy on any theory that assumes the existence of positive energy only. To calculate the energy in terms of the electric and magnetic forces we must distinguish between the forces due to positive charges which we may call $\overset{+}{\mathbf{E}}$ and $\overset{+}{\mathbf{H}}$, and those due to negative charges, which we may call $\overset{-}{\mathbf{E}}$ and $\overset{-}{\mathbf{H}}$. In this case the total energy of the distribution is

$$\frac{1}{8\pi}\int\int_{\infty}\int \{(\mathbf{E}^2+\mathbf{H}^2)-\frac{1}{k}(\mathbf{g}^2+\mathbf{h}^2)\}\,d\tau,$$

$$=\frac{1}{8\pi}\int\int_{\infty}\int \{(1-G)\,(\overset{+}{\mathbf{E}}^2+\overset{+}{\mathbf{H}}^2)+(\overset{-}{\mathbf{E}}^2+\overset{-}{\mathbf{H}}^2)+2(\overset{+}{\mathbf{E}}\cdot\overset{-}{\mathbf{E}}+\overset{+}{\mathbf{H}}\cdot\overset{-}{\mathbf{H}})\}\,d\tau,$$

where we have exactly the same distribution of total energy in each element; and we have made a change only in assuming that the whole positive charge and whole negative charge have each a much larger amount of positive energy than the total energy of the system, but that negative mutual energy of the two distributions more than neutralizes their positive energy in all volume elements outside the mass, and fails to do so only within the mass.

Stresses in the Gravitational Field. — We have by no means yet exhausted the store of interesting facts connected with this negative

energy. For the similarity of the gravitational equations to the electromagnetic tells us that we may expect gravitational energy to have inertia, like that of electromagnetic energy, and also that we may expect to find stresses in the gravitational field like those of the electromagnetic field. By the word "stress" I mean, of course, "force per unit area," but it must be thoroughly understood that "force" is defined merely as "that which produces acceleration in anything possessing inertia," so that the force must be thought of as accelerating the electromagnetic or gravitational energy of the medium rather than the medium itself, which is incapable of motion.

From the exact similarity of the gravitational field around a distribution of moving masses to the electromagnetic field around a similar distribution of charges constrained to move in similar paths, and from the fact that the mechanical forces of the fields in one case are exactly opposite to those in the other, we see that the stresses in the gravitational field are exactly opposite to those of the electromagnetic field. This gives us the following set of stresses: a pressure along the lines of $\mathbf{g}$ with a tension across them, each of intensity $\frac{\mathbf{g}^2}{8\pi k}$, equal to the negative energy per unit volume of the vector $\mathbf{g}$; and a pressure along the lines of $\mathbf{h}$ with a tension across them, each of intensity $\frac{\mathbf{h}^2}{8\pi k}$, equal to the negative energy per unit volume of the vector $\mathbf{h}$. These results look, at first sight, impossible, especially when we notice that these stresses have such enormous values as six hundred tons per square centimeter at the surface of the earth, and forty thousand tons per square centimeter on the sun; because it appears as if the forces in a static distribution would be in a state of unstable equilibrium, in which the least disturbance would cause the lines of force to crumple into a hopeless tangle, but it will appear presently that this is not the case at all.

Negative Inertia. — To see why not, we may again consider our corresponding electrical case, and consider the similarity of the vectors $\mathbf{E}\times\mathbf{H}$ and $\mathbf{g}\times\mathbf{h}$, which have exactly similar lines with proportional intensities at different points, and, what is most important, the same direction in every case. And, as Lorentz [11] proves, in space containing no matter, the force due to the stresses on the surface of any element $d\tau$ in an electromagnetic field is

$$\left(\frac{1}{4\pi c^2}\frac{\partial(\mathbf{E}\times\mathbf{H})}{\partial t}\,d\tau\right).$$

[11] Lorentz, Theory of Electrons, chapter I, p. 23 et seq.

Therefore we see that in empty space in the gravitational field we have force due to the stresses on the surface of the element $d\tau$ equal to

$$\left(-\frac{1}{4\pi c^2 k}\frac{\partial(\mathbf{g}\times\mathbf{h})}{\partial t}\,d\tau\right).$$

But the vector $\mathbf{g}\times\mathbf{h}$ represents the rate of flow of negative energy per unit volume, and hence it appears that, if the stresses in the electromagnetic field accelerate positive energy with its ordinary inertia, the stresses in the gravitational field, which act against the acceleration, must accelerate negative energy with its *negative inertia*. It is now evident that since the system of stresses we have in the gravitational field would give unstable equilibrium in a set of lines of force with positive inertia, they give stable equilibrium in a similar set with negative inertia. And furthermore, it is evident that if we reverse the signs of both stress and inertia, as we do in changing from electromagnetic to gravitational conditions, the motions in free space of the negative gravitational energy will be exactly like those of the corresponding positive electromagnetic energy.

To see what effect this negative inertia of the gravitational field has upon the motion of any mass, we may consider its kinetic energy for different velocities. For a small sphere, either positively or negatively charged, and subject to the deformations required by the relativity principle, the electromagnetic energy for any velocity may be shown to be $\triangle mc^2\beta^{-1}$, where $\triangle m$ is the electromagnetic inertia and $\triangle mc^2$ the electrostatic energy when it is at rest. If we now assume that matter is made up of minute electric charges, we see that if these charges are near enough to each other to have their fields of energy overlap to any appreciable extent, then if m is the sum of the inertias of the particles in a body when separated, and hence the gravitational mass of the whole body, the inertia of the whole body may be a different quantity m'. If the overlapping is that of fields of charges of opposite sign, it will make m' less than m, but if it is that of fields from those of the same sign, it will make m' greater than m. But if unbalanced effects of this sort could occur, it is evident that we could expect equal masses of different substances to have different inertias, and since these different inertias have never been observed, we may neglect such effects and say that the electromagnetic inertia of any body is the sum of the inertias of its component charges. This has been tacitly assumed in developing the theory, and we now see that it is a reasonable assumption.[12]

[12] This assumption is inconsistent with the idea that positive electricity may be freely penetrable to negative electrons, as has sometimes been sup-

Since the body as a whole is subject to the same deformations with change of velocity, we may prove that if its gravitational energy when at rest is $-m_g c^2$, its gravitational energy when in motion will be $-m_g c^2 R^{-1}$, so that the effect of the negative inertia of the gravitational energy will be to reduce the total inertia in the ratio $1 : (1-m_g/m)$.

This result is so important that it is well to look at it also from the point of view of inertia as the property of resisting acceleration. In the case of a positively charged body which is accelerated, we see that every element of charge in it will radiate forces which have at all points outside the element components in the direction opposite to that of the acceleration. Therefore, during a constant acceleration each element will be acted upon by forces radiated from other elements which will hold it back by an amount proportional to the acceleration. But in the gravitational case the corresponding forces will act in the direction of the acceleration, and will therefore help, instead of hinder, the action of the accelerating force. And it will not be noticed that since these forces are inversely proportional to the distance between the elements, the inertias thus obtained will be inversely proportional to any dimension in two similar but unequal bodies with equal charges or masses. We also see that if we extend these considerations to cases of variable accelerations, the electromagnetic effects introduce the tendency to increase the rate of change of acceleration which has the effect of a force in the direction of this rate of the order of

$$\frac{e^2}{c^3}\frac{d\mathbf{a}}{dt}$$

and independent of the size of the body. Much use has been made of this in the theory of emission and absorption of light. We see, too, that the gravitational energy will introduce a similar force, whose ratio to that of the electromagnetic energy is not m_g/m.

It appears now, on account of these changes in the inertia of bodies by the gravitational energy, that we must modify the statements made above about the lack of radiation of negative energy from bodies moving under the action of gravitation. But the modification is not so great as might be supposed, because the case of the symmetrical body rotating on its axis of symmetry still involves no radiation, and the radiation in other cases is still less than what could be expected with other theories. Hence we see that in spite of this modification of the theory, the arguments for its preference over all others are still valid.

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posed; but it is only one of many objections to this idea, so that the inconsistency need not cause much doubt as to the truth of the assumption.

To form an idea of the extent to which the inertia of a large mass is diminished by the gravitational energy, we may calculate the ratio m_g/m for a sphere of uniform density ρ and radius a. In such a case, at all external points whose distance is r the vector $\mathbf{g}$ has the intensity km/r^2, while at internal points its intensity is kmr/a^3. The energy is now

$$-m_g c^2 = -\frac{1}{8\pi k}\int_0^\infty 4\pi r^2 \mathbf{g}^2 dr,$$

which is readily shown to be

$$-\frac{3}{5}\frac{m^2 k}{a}.$$

Hence we see that

$$\frac{m_g}{m} = \frac{3}{5}\frac{mk}{c^2 a} = \frac{4\pi}{5}\cdot\frac{k\rho a^2}{c^2}.$$

For a sphere of homogeneous density and of size and mass equal to that of the sun, we have

$$m = 2.0 \times 10^{33},\ a = 7.0 \times 10^{10},\ k = 6.5 \times 10^{-8},\ c^2 = 9.0 \times 10^{20},$$

so that

$$\frac{m_g}{m} = 1.2 \times 10^{-6}.$$

The inertia of the sun is therefore diminished by about one part in a million by the gravitational energy it possesses.

Consequences of Negative Inertia. — Small as it is, this minute diminution of inertia shows clearly the way to the ultimate condition of the universe. For we may imagine the sun and all the rest of the stars radiating their heat away as they drift through space, and having the supply renewed only by occasional collisions, each one of which combines the masses of the colliding bodies into a mass as great as both together, but with an inertia, after the heat of collision is lost,[13] not as great as the sum of those of the original masses. Thus a time will come when one of the masses formed by this process will have actually less inertia than one of the parts of which it is made up, and will readily be accelerated to enormous velocities by the least attractions. And at last there will be formed a tremendous mass, of the order of 10^9 times the mass of the sun, whose inertia will be nega-

[13] Because heat is a form of motion of bodies with electromagnetic mass, we may consider it as electromagnetic energy, with inertia like that of any other such energy.

tive;[14] so that instead of being drawn towards its neighbors in space by their attractions it will appear to avoid them as if it were repelled, while they, being drawn towards it by its enormous attraction, will pursue it as it flees from them.

The more sluggish of them will soon be left far behind, but the livelier ones, with the least inertia, will draw nearer and nearer, until at last one will overtake it, *hit it from behind and slow it down.* By this process its negative inertia will be further increased, until it has collected to itself all matter with positive inertia within reach.

This process will be repeated in many different parts of the universe until at last all matter will be collected in enormous masses with negative inertia. They will then act as if they all had positive inertia but repelled each other, and, therefore, there will be no more collisions. But their heat will be lost by radiation, with no chance to renew the supply, and at last, if the present laws of nature still hold, the universe will be cold and dead forever.

Are all Forces Electromagnetic? — And now that we see that the reason gravitational forces obey the principle of relativity is the fact that they are really electromagnetic forces, and now we know from the Michelson-Morley experiment that all internal stresses in even the most rigid bodies obey the same principle, we may say that we have at last some evidence to show that all such stresses are of electromagnetic origin. And we may now say that the principle of relativity applies to all the laws of nature because it applies to all the laws of electromagnetic phenomena, and because all the laws of nature are laws of electromagnetic phenomena.

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[14] It might be supposed, at first sight, that just when the inertia became zero in the process of losing electromagnetic energy, the mass might have an infinite acceleration. But it will be noticed that the tendencies to resist change of acceleration, not being in the ratio $-m_0/m$, will not be equal and opposite, so that no such effects will occur.